BASIC COMPARISON TEST III (OF III)

The essence of the basic comparison test is to compare a given series $\sum a_n$ to another series $\sum b_n$, which we know converges or diverges. In order to know how $\sum b_n$ behaves, we should try to choose the underlying sequence (b_n) as simple as possible. On the other hand it should be similar to the sequence (a_n) so that we are still able to relate a_n to b_n. Making a good choice of the comparison series requires some practice.

Example 3.1: Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{(2n+1)^4}$. If we just consider the highest power of n then the sequence $\displaystyle\left(\frac{n^2}{(2n+1)^4}\right)_{n\geq 1}$ behaves similarly to the sequence $\displaystyle\left(\frac{n^2}{n^4}\right)_{n\geq 1}$, which is the sequence $\displaystyle\left(\frac{1}{n^2}\right)_{n\geq 1}$. Also we have

$$0 \leq \frac{n^2}{(2n+1)^4} < \frac{1}{n^2}, \quad \text{for all } n \neq 1.$$

This is a consequence of $n^4 < (2n+1)^4$, for all $n \geq 1$. Furthermore $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, since it is a p-series with $p = 2$, (for details refer to the handouts on p-Series). So the first part of the basic comparison test implies that

$$\sum_{n=1}^{\infty} \frac{n^2}{(2n+1)^4} \quad \text{also converges.}$$

Example 3.2: Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5\sqrt{n}-2}$. For large n the sequence $\displaystyle\left(\frac{1}{5\sqrt{n}-2}\right)_{n\geq 1}$ behaves like the sequence $\displaystyle\left(\frac{1}{\sqrt{n}}\right)_{n\geq 1}$. Since $0 < 4\sqrt{n}-2$, for all $n \geq 1$, we have $\sqrt{n} < 5\sqrt{n}-2$, for all $n \geq 1$. Consequently

$$0 \leq \frac{1}{5\sqrt{n}-2} < \frac{1}{\sqrt{n}}, \quad \text{for all } n \geq 1.$$

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But $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a p-series, with $p = \frac{1}{2}$. Thus this series diverges to infinity. Observe that the basic comparison test cannot be applied in this situation. Our misfortune (but not mistake, because how could we have known) was that we have chosen a series which was too simple for the comparison. That means we have to reconsider our choice.

Observe that for large n the sequence $\left(\frac{1}{5\sqrt{n}-2}\right)_{n\geq 1}$ also behaves similar to the sequence $\left(\frac{1}{5\sqrt{n}}\right)_{n\geq 1}$. Furthermore we have

$$0 \leq \frac{1}{5\sqrt{n}} < \frac{1}{5\sqrt{n}-2}, \quad \text{for all } n \geq 1,$$

and the series $\sum_{n=1}^{\infty} \frac{1}{5\sqrt{n}} = \frac{1}{5}\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges to infinity. Thus the second part of the basic comparison test implies that $\sum_{n=1}^{\infty} \frac{1}{5\sqrt{n}-2}$ diverges to infinity as well.

Remark 3.3: Given a series it sometimes might be cumbersome to find a suitable second series to apply the basic comparison test. This is particularly true if the series is close to the borderline between convergence and divergence. In such cases the limit comparison test might be a quicker solution to the problem. In Remark 2.2 on the handout Limit Comparison Test II we discuss this issue further.