GEOMETRIC SERIES II (OF II)

Limit of a Geometric Series

The limit of a geometric series is fully understood and depends only on the position of the number x on the real line.

$$(1)\text{ If } x \leq -1, \quad \text{then} \quad \sum_{n=0}^{\infty} x^n \text{ does not exist.}$$

$$(2)\text{ If } |x| < 1, \quad \text{then} \quad \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

$$(3)\text{ If } x \geq 1, \quad \text{then} \quad \sum_{n=0}^{\infty} x^n = \infty.$$

Observe that every real number x falls into exactly one of the three cases $x \leq -1$, $|x| < 1$ (or equivalently, $-1 < x < 1$) and $x \geq 1$. In particular for every x we understand the limit of the corresponding geometric series. Also note that the geometric series only converges if $|x| < 1$, and in this case we know that it converges towards $\frac{1}{1-x}$. Furthermore if $x \geq 1$, the geometric series diverges to infinity. This is clear as we are adding up increasingly larger numbers and thus surpass any boundary.

Example 2.1: Let $x = 1$. In this case we have $x \geq 1$, and so

$$\sum_{n=0}^{\infty} 1^n = \infty,$$

that is, this geometric series diverges to infinity. We can verify this result by taking a look at the sequence of partial sums $s_0 = 1$, $s_1 = 2$, $s_2 = 3$, $s_3 = 4$, $s_4 = 5$ and so on. (See also Example 1.1 on the handout Geometric Series I). Clearly this sequence tends to infinity.

Example 2.2 Let $x = -2$. In this case we have $x \leq -1$, and so

$$\sum_{n=0}^{\infty} (-2)^n \quad \text{does not exist.}$$

Material developed by the Department of Mathematics & Statistics, NUIM and supported by www.ndlr.com.

Here the sequence of partial sums is given by $s_0 = 1$, $s_1 = -1$, $s_2 = 3$, $s_4 = -5$, $s_5 = 11$ and so on. (See also Example 1.2 on the handout Geometric Series I). Note how the sequence elements alter between positive and negative integers making the existence of a limit impossible.

Example 2.3 Let $x = \dfrac{1}{2}$. In this case we have $|x| < 1$, and so

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2,$$

that is, this geometric series converges to 2. Recall from Example 1.3 on the handout Geometric Series I, that the sequence of partial sums is given by $s_0 = 1$, $s_1 = \frac{3}{2}$, $s_2 = \frac{7}{4}$, $s_3 = \frac{15}{8}$, $s_4 = \frac{31}{16}$ and so on. One can check that $s_n = \frac{2^{n+1}-1}{2^n}$, for all $n \geq 0$. Note that this sequence of partial sums really tends towards 2.

Example 2.4 Let us find the limit of the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{3^n}$. This series is a geometric series. However it is important to realize that $\frac{1}{3^n} = \left(\frac{1}{3}\right)^n$, which means that $x = \frac{1}{3}$ and NOT $x = 3$. Hence $|x| < 1$ and we get

$$\sum_{n=0}^{\infty} \frac{1}{3^n} = \frac{1}{1-\frac{1}{3}} = \frac{3}{2},$$

that is, the series converges to $\dfrac{3}{2}$.

Example 2.5 Let us find the limit of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$. This series is almost the geometric series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{3^n}$ with the only difference that n starts at 1 rather than 0. Since $\dfrac{1}{3^0} = 1$ we can say

$$\sum_{n=1}^{\infty} \frac{1}{3^n} = \left(\sum_{n=0}^{\infty} \frac{1}{3^n}\right) - 1$$

From the previous example we know that $\displaystyle\sum_{n=0}^{\infty} \frac{1}{3^n} = \frac{3}{2}$ and thus

$$\sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{3}{2} - 1 = \frac{1}{2},$$

that is, the series converges to $\dfrac{1}{2}$.