LIMIT COMPARISON TEST II (OF II)

Example 2.1: Consider the series $\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$. Then the underlying sequence is

$$a_n = \frac{1}{1+\sqrt{n}}, \quad \text{for all } n \geq 1.$$

Also note that all sequence elements are positive. Again we need to look for an appropriate sequence b_n. We observe that the numerator of a_n is constantly one, while the denominator is dominated by $\sqrt{n}$. Hence we should try the positive sequence

$$b_n = \frac{1}{\sqrt{n}}, \quad \text{for } n \geq 1.$$

Then

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{1+\sqrt{n}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{1+\sqrt{n}} = \lim_{n\to\infty} \frac{1}{(1/\sqrt{n})+1} = 1 > 0.$$

Furthermore

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}} = \infty.$$

This follows as the given series is a p-series with $p = 1/2$. Hence part (2) of the limit comparison test now implies that

$$\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}} = \infty.$$

Remark 2.2: Often we can apply both the limit comparison test and basic comparison test to a series. However it is important to emphasize a subtle difference between both tests. Let us reconsider the series $\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$ of Example 2.1. This time we show the divergence of this series using the basic comparison test.

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Let $n \geq 3$. Then $n - \sqrt{n} = \sqrt{n}(\sqrt{n} - 1) > \sqrt{3}(\sqrt{3} - 1) = 3 - \sqrt{3} > 1$. Hence we have shown that $n > 1 + \sqrt{n}$, for all integers $n \geq 3$. Therefore

$$\frac{1}{n} < \frac{1}{1 + \sqrt{n}}, \quad \text{for all } n \geq 3.$$

As the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges to infinity, the basic comparison test now implies that $\sum_{n=1}^{\infty} \frac{1}{1 + \sqrt{n}}$ diverges to infinity. (For more details refer to the handouts on the basic comparison test.)

At first sight it appears that the basic comparison test gives a much quicker answer than the limit comparison test did in Example 2.1. However that is only true if we already have enough insight to realize that we need to choose the harmonic series for the basic comparison test to work. It is not clear why we would pick the harmonic series in the first place.

In Example 2.1 there was a clear reason why we picked the sequence $\frac{1}{\sqrt{n}}$ for comparison. If we attempted the basic comparison test with that obvious choice, then we would see that

$$\frac{1}{1 + \sqrt{n}} < \frac{1}{\sqrt{n}}, \quad \text{for all } n \geq 1.$$

However, in this situation the basic comparison test cannot be applied, as the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges to infinity. That means in order to use the basic comparison test we would have to find a different sequence. By the time we have worked out which other choice may work, we might have found the answer using the slightly longer limit comparison test.