Limit Comparison Test

Let $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ be two positive sequences such that $\lim\limits_{n\to\infty} \frac{a_n}{b_n}$ exists. Set

$$L := \lim_{n\to\infty} \frac{a_n}{b_n}.$$

(Note that L is a non-negative real number or $L = \infty$.) Then

(1) If $L < \infty$ and $\sum\limits_{n=0}^{\infty} b_n$ converges, then $\sum\limits_{n=0}^{\infty} a_n$ also converges.

(2) If $L > 0$ and $\sum\limits_{n=0}^{\infty} b_n$ diverges to ∞, then $\sum\limits_{n=0}^{\infty} a_n$ diverges to ∞.

Remark 1.1: So if we want to learn something about the series $\sum a_n$, the limit comparison test suggests that we should look for an appropriate series $\sum b_n$, where the underlying sequences (a_n) and (b_n) behave similarly in the sense that $\lim\limits_{n\to\infty} \frac{a_n}{b_n}$ exists. Then the convergent behaviour of the series $\sum a_n$ coincides with that of the series $\sum b_n$, that is, if we understand how $\sum b_n$ behaves then we understand $\sum a_n$.

Example 1.2: Consider the series $\sum\limits_{n=1}^{\infty} \frac{2\sqrt{n}+3}{n^2\sqrt{n}+4\sqrt{n}}$. Then the underlying sequence is

$$a_n = \frac{2\sqrt{n}+3}{n^2\sqrt{n}+4\sqrt{n}}, \quad \text{for all } n \geq 1,$$

and all sequence elements are positive. In order to apply the limit comparison test successfully we need to find a second sequence $(b_n)_{n\geq 1}$ that behaves similarly to $(a_n)_{n\geq 1}$ and where, in addition, we know whether the series $\sum b_n$ converges or diverges.

Note that as n grows bigger the numerator of a_n is dominated by $2\sqrt{n}$. (That means the influence of the term $2\sqrt{n}$ on the numerator increases, whereas the influence of the term 3 remains the same.) Looking at the

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denominator we see that it is dominated by $n^2\sqrt{n}$. (Here the remaining term $4\sqrt{n}$ also increases with an increasing n, but not as fast as $n^2\sqrt{n}$.) This reasoning tells us that, for large n, the element a_n is similar to $\frac{2\sqrt{n}}{n^2\sqrt{n}}$ which equals to $\frac{2}{n^2}$, and thus we choose

$$b_n = \frac{2}{n^2}, \quad \text{for } n \geq 1.$$

We see that $(b_n)_{n\geq 1}$ is a positive sequence, and

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2\sqrt{n}+3}{n^2\sqrt{n}+4\sqrt{n}} \cdot \frac{n^2}{2} = \lim_{n\to\infty} \frac{2n^2\sqrt{n}+3n^2}{2n^2\sqrt{n}+8\sqrt{n}}$$
$$= \lim_{n\to\infty} \frac{2+\frac{3}{\sqrt{n}}}{2+\frac{8}{n^2}} = \frac{2}{2} = 1.$$

In particular L exists and is finite, (hence our choice of b_n was good). Furthermore

$$\sum_{n=1}^{\infty} \frac{2}{n^2} = 2 \cdot \sum_{n=1}^{\infty} \frac{1}{n^2}$$

is a p-series where $p = 2$. Therefore it converges. (For more details refer to the handouts on p-series.) Now part (1) of the limit comparison test implies that $\sum_{n=1}^{\infty} \frac{2\sqrt{n}+3}{n^2\sqrt{n}+4\sqrt{n}}$ converges as well.

Example 1.3: Consider the series $\sum_{n=1}^{\infty} \frac{1}{2^n(n+1)}$. Here the underlying sequence is $a_n = \frac{1}{2^n(n+1)}$, for all $n \geq 1$. Let $b_n = \frac{1}{2^n}$, for $n \geq 1$. Then $(b_n)_{n\geq 1}$ is a positive sequence, and

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{2^n(n+1)} \cdot \frac{2^n}{1} = \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

In particular L exists and is finite. Finally

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$$

is a geometric series where $x = \frac{1}{2}$, and thus converges (see also Example 1.3 on the handout Geometric Series I). By part (1) of the limit comparison test we can now conclude that $\sum_{n=1}^{\infty} \frac{1}{2^n(n+1)}$ converges.