Limit Examples 2

In this handout we will look at rational functions of the form

$$f(x) = \frac{g(x)}{h(x)}$$

where $g(x)$ and $h(x)$ are polynomials.
If $f(a)$ does not exist, then $h(a) = 0$ and so $x - a$ must be a factor of $h(x)$. We will exploit this fact when calculating limits involving rational functions. The trick is to factorise the numerator and/or the denominator in the hope that we can cancel the $x - a$ term.

Example 1
Find

$$\lim_{x \to 1} \frac{x^2 - 9x + 8}{x - 1}$$

We begin by substituting $x = 1$ into the function to see if it will become undefined. We notice that if we let $x = 1$ in the denominator (the bit underneath the line), we would be dividing by 0. Therefore we have a problem. In situations like this, all we really can do is factorise the numerator and/or the denominator and hope that we can cancel any $x - 1$ terms.

$$\begin{aligned} \lim_{x \to 1} \frac{x^2 - 9x + 8}{x - 1} &= \lim_{x \to 1} \frac{(x-8)(x-1)}{(x-1)} \\ &= \lim_{x \to 1} x - 8. \end{aligned}$$

Just like we hoped, we were able to cancel a factor of $(x - 1)$ from our problem. We can now calculate the limit because $x - 8$ is a polynomial and $\lim_{x \to a} p(x) = p(a)$ for all polynomials $p(x)$. We have:

$$\begin{aligned} \lim_{x \to 1} x - 8 &= 1 - 8 \\ &= -7. \end{aligned}$$

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Example 2
Find

$$\lim_{x \to 3} \frac{x^2 - x - 6}{x^2 - 3x}$$

Notice that if we let $x = 3$ in the denominator, we would be dividing by 0. Therefore just like in Example 1, all we really can do is factorise the numerator and denominator in the hope that we can cancel any $x - 3$ terms from the denominator.

$$\begin{aligned} \lim_{x \to 3} \frac{x^2 - x - 6}{x^2 - 3x} &= \lim_{x \to 3} \frac{(x+2)(x-3)}{x(x-3)} \\ &= \lim_{x \to 3} \frac{(x+2)}{x}. \end{aligned}$$

Just like we hoped, we were able to cancel a factor of $(x - 1)$ from our problem. We can now calculate the limit because $\frac{x+2}{x}$ is a rational function where the denominator is not zero at $x = 3$. We have:

$$\begin{aligned} \lim_{x \to 3} \frac{x+2}{x} &= \frac{3+2}{3} \\ &= \frac{5}{3}. \end{aligned}$$

See "Limit Examples 3" for examples of where this technique does not work.

Try the following exercises for practice.

Calculate the following limits:

(a) $\displaystyle\lim_{x \to 3} \frac{x^2 - 5x + 6}{x - 3}$ (b) $\displaystyle\lim_{x \to -5} \frac{x^2 + x - 20}{x + 5}$ (c) $\displaystyle\lim_{x \to -3} \frac{2x^2 - 6x - 36}{x + 3}$

(d) $\displaystyle\lim_{x \to 9} \frac{x^2 - 8x - 9}{x^2 - 81}$ (e) $\displaystyle\lim_{x \to 4} \frac{x - 4}{x^2 + 2x - 24}$

Solutions
(a) 1 (b) -9 (c) -18 (d) $\frac{5}{9}$ (e) $\frac{1}{10}$