Properties of Limits

In this hand-out we will look at the properties of limits. These properties allow us to manipulate and solve limit problems.

Suppose $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ where $a, L, M \in \mathbb{R}$. Then:

1.
$$\lim_{x\to a} x = a$$

2.
$$\lim_{x\to a} c = c \quad \text{where c is a constant.}$$

3.
$$\lim_{x\to a}(f(x) + g(x)) = L + M$$

4.
$$\lim_{x\to a}(f(x) - g(x)) = L - M$$

5.
$$\lim_{x\to a} f(x).g(x) = L.M$$

6.
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M} \quad \text{if } M \neq 0.$$

7.
$$\lim_{x\to a} k.f(x) = k.L \quad \text{where } k \text{ is a constant.}$$

8.
$$\lim_{x\to a}(f(x))^p = L^p \quad \text{where } p \text{ is a positive integer.}$$

9. If p is a positive integer then
$$\lim_{x\to a}(f(x))^{\frac{1}{p}} = \begin{cases} L^{\frac{1}{p}} & , \quad p \text{ odd} \\ L^{\frac{1}{p}} & , \quad p \text{ even and } L \geq 0 \end{cases}$$

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Using the above properties it can be shown that $\lim_{x\to a} p(x) = p(a)$ where $p(x)$ is a polynomial and $a \in \mathbb{R}$.

We will now calculate the following limits explicitly, using the properties above.

Example 1

$$\begin{aligned}
\lim_{x\to 2} x^2 + 3x - 2 &= \lim_{x\to 2} x^2 + \lim_{x\to 2} 3x - \lim_{x\to 2} 2 && \text{using properties 3 and 4.} \\
&= \left(\lim_{x\to 2} x\right)^2 + 3\lim_{x\to 2} x - \lim_{x\to 2} 2 && \text{using properties 7 and 8.} \\
&= (2)^2 + 3(2) - 2 && \text{using properties 1 and 2.} \\
&= 4 + 6 - 2 \\
&= 8.
\end{aligned}$$

Note, since $p(x) = x^2 + 3x - 2$ is a polynomial, $\lim_{x\to 2} p(x) = p(2) = 8$.

Example 2

$$\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{2x+4}}{\cos x} &= \frac{\lim_{x\to 0} \sqrt{2x+4}}{\lim_{x\to 0} \cos x} && \text{using property 6.} \\
&= \frac{\lim_{x\to 0} (2x+4)^{\frac{1}{2}}}{\lim_{x\to 0} \cos x} \\
&= \frac{(\lim_{x\to 0} 2x+4)^{\frac{1}{2}}}{\lim_{x\to 0} \cos x} && \text{using property 9.} \\
&= \frac{(2\lim_{x\to 0} x + \lim_{x\to 0} 4)^{\frac{1}{2}}}{\lim_{x\to 0} \cos x} && \text{using properties 3 and 7.} \\
&= \frac{(2(0)+4)^{\frac{1}{2}}}{1} && \text{using properties 1 and 2.} \\
&= (4)^{\frac{1}{2}} \\
&= \sqrt{4} \\
&= 2.
\end{aligned}$$

(Note that $\lim_{x\to 0} \cos x = 1$.)

Using the properties of limits overleaf, evaluate the follow limits explicitly as in the examples above.

(a) $\lim_{x\to 4} 4x^3 - 5x + 2$ (b) $\lim_{x\to -2} \frac{2x^3+5}{\sqrt{4x+24}}$ (c) $\lim_{x\to \frac{\pi}{3}} \frac{x+\pi}{\cos x}$

Solutions

(a) 238 (b) $-\frac{11}{4}$ (c) $\frac{8\pi}{3}$