The Ratio Test

Let $\sum a_n$ be a series with non-zero terms a_n. Moreover suppose that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lambda,$$

(that is, $\left| \frac{a_{n+1}}{a_n} \right|$ tends to the value λ, as n tends to infinity. Here λ is allowed to be any finite number as well as ∞.) Then

(a) If $\lambda < 1$, then $\sum a_n$ converges.

(b) If $\lambda > 1$, then $\sum a_n$ does not converge.

(c) If $\lambda = 1$, then the ratio test in inconclusive.

Remark 1.1: That means we need to determine $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$ and if this limit exists and is different from one, then we can make a statement about the convergence of the given series $\sum a_n$. The ratio test is very effective with factorials and combination of factorials and powers.

Example 1.2: Consider the series $\sum_{n=1}^{\infty} \frac{1}{n!}$. Then the underlying sequence is

$$a_n = \frac{1}{n!}, \quad \text{for all } n \geq 1,$$

and clearly all sequence elements are non-zero (which is one of the conditions that need to be satisfied to apply the ratio test). Furthermore we need to understand $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$. We have

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \frac{n!}{(n+1)!} = \frac{1}{n+1}.$$

So $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$, and thus $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges.

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Example 1.3: Consider the series $\sum_{n=1}^{\infty} \frac{n!}{10^n}$. Then

$$a_n = \frac{n!}{10^n}, \quad \text{for all } n \geq 1,$$

and all sequence elements are non-zero. Furthermore we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!}{10^{n+1}} \cdot \frac{10^n}{n!} = \frac{(n+1)!}{n!} \cdot \frac{10^n}{10^{n+1}} = \frac{n+1}{10}$$

Hence $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n+1}{10} = \infty > 1$, and so $\sum_{n=1}^{\infty} \frac{n!}{10^n}$ does not converge, by the ratio test. As all elements of the underlying sequence a_n are positive we conclude that the series diverges to plus infinity.

Example 1.4: Consider the series $\sum_{n=1}^{\infty} \frac{1}{2n+1}$. Then

$$a_n = \frac{1}{2n+1}, \quad \text{for all } n \geq 1,$$

and all sequence elements are non-zero. Furthermore we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{2(n+1)+1} \cdot \frac{2n+1}{1} = \frac{2n+1}{2n+3} = \frac{2+\frac{1}{n}}{2+\frac{3}{n}}$$

So $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2+\frac{1}{n}}{2+\frac{3}{n}} = 1$, which means that the ratio test is inconclusive. Nevertheless one can still determine that the series $\sum_{n=1}^{\infty} \frac{1}{2n+1}$ diverges to infinity, for instance, by means of the comparison test (see Example 2.2 on the handout Basic Comparison Test II).

Example 1.5: The alternating series $\sum_{n=-10}^{\infty} \left(\frac{-1}{100}\right)^n$ converges, (see Example 3.2 on the handout Alternating Series III). We can also apply the ratio test to show convergence. The underlying sequence is given by $a_n = \left(\frac{-1}{100}\right)^n$, for all $n \geq -10$, and all sequence elements are non-zero. Furthermore we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(\frac{-1}{100})^{n+1}}{(\frac{-1}{100})^n}\right| = \left|\frac{-1}{100}\right| = \frac{1}{100}.$$

So $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{100} < 1$, and thus $\sum_{n=-10}^{\infty} \left(\frac{-1}{100}\right)^n$ converges.