Rationalising Technique

In other handouts (Limit Examples 1,2 and 3) we see that when calculating the $\lim_{x\to a} f(x)$ of a rational function, the function is often undefined at $x = a$. The first thing to try is factorising, in the hope that any $x - a$ terms will cancel. When our function is not rational, it is sometimes possible to create a common factor. One such method is the rationalising technique and examples of this technique are described below.

Example 1
Find

$$\lim_{x\to 0} \frac{\sqrt{2+x}-\sqrt{2}}{x}.$$

We begin by substituting $x = 0$ into the function to see if it will become undefined. We notice that if we let $x = 0$ in the denominator, we would be dividing by 0 and thus we have a problem. We also notice that there is no obvious factorisation we could perform to eliminate this problem. We use a trick called rationalising to get around this issue. The rationalising technique for evaluating limits is based on multiplication by a convenient form of 1. We multiply the numerator and denominator by the conjugate of the numerator in this case. (The conjugate of $a + b$ is $a - b$.)

$$\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{2+x}-\sqrt{2}}{x} &= \lim_{x\to 0} \frac{(\sqrt{2+x}-\sqrt{2})}{(x)} \frac{(\sqrt{2+x}+\sqrt{2})}{(\sqrt{2+x}+\sqrt{2})} \\
&= \lim_{x\to 0} \frac{(\sqrt{2+x})^2+\sqrt{2}\sqrt{2+x}-\sqrt{2}\sqrt{2+x}-(\sqrt{2})^2}{(x)(\sqrt{2+x}+\sqrt{2})} \\
&= \lim_{x\to 0} \frac{2+x-2}{(x)(\sqrt{2+x}+\sqrt{2})} \\
&= \lim_{x\to 0} \frac{x}{(x)(\sqrt{2+x}+\sqrt{2})} \\
&= \lim_{x\to 0} \frac{1}{\sqrt{2+x}+\sqrt{2}} \\
&= \frac{1}{\sqrt{2+0}+\sqrt{2}} \\
&= \frac{1}{2\sqrt{2}}.
\end{aligned}$$

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Example 2

$$\begin{aligned}
\lim_{x\to 4}\frac{2x-8}{\sqrt{x+5}-3} &= \lim_{x\to 4}\frac{(2x-8)}{(\sqrt{x+5}-3)}\frac{(\sqrt{x+5}+3)}{(\sqrt{x+5}+3)} \\
&= \lim_{x\to 4}\frac{(2x-8)(\sqrt{x+5}+3)}{(\sqrt{x+5})^2+3\sqrt{x+5}-3\sqrt{x+5}-9} \\
&= \lim_{x\to 4}\frac{(2x-8)(\sqrt{x+5}+3)}{x-4} \\
&= \lim_{x\to 4}\frac{2(x-4)(\sqrt{x+5}+3)}{x-4} \\
&= \lim_{x\to 4}2(\sqrt{x+5}+3) \\
&= 2(\sqrt{4+5}+3) \\
&= 12.
\end{aligned}$$

Try the following exercises for practice:

(a)

$$\lim_{x\to 0}\frac{\sqrt{5+x}-\sqrt{5}}{x}$$

(b)

$$\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}$$

(c)

$$\lim_{x\to 3}\frac{\sqrt{x+1}-2}{x-3}$$

(d)

$$\lim_{x\to 1}\frac{x-1}{\sqrt{x+3}-2}$$

(e)

$$\lim_{x\to -1}\frac{\sqrt{x^2+8}-3}{x+1}$$

Solutions

(a) $\frac{1}{2\sqrt{5}}$ (b) $\frac{1}{2}$ (c) $\frac{1}{4}$ (d) 4 (e) $-\frac{1}{3}$