

Sine Limit Examples

We begin by introducing the following two rules:

$$(i) \quad \lim_{x \rightarrow 0} \frac{\sin kx}{kx} = 1 \quad \text{and} \quad (ii) \quad \lim_{x \rightarrow 0} \frac{kx}{\sin kx} = 1 \quad \text{for } k \in \mathbb{R} \text{ and } k \neq 0.$$

For example we can say:

$$\lim_{x \rightarrow 0} \frac{\sin 5x}{5x} = 1 \quad \text{or} \quad \lim_{\theta \rightarrow 0} \frac{3\theta}{\sin 3\theta} = 1 \quad \text{or} \quad \lim_{x \rightarrow 0} \frac{\sin\left(-\frac{1}{2}x\right)}{-\frac{1}{2}x} = 1$$

Example 1

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin 4x}{x} &= \lim_{x \rightarrow 0} \frac{4 \sin 4x}{4x} && \text{Multiplying top and bottom by 4} \\ &= 4 \lim_{x \rightarrow 0} \frac{\sin 4x}{4x} \\ &= 4(1) && \text{Applying rule (i) above} \\ &= 4. \end{aligned}$$

Example 2

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{3x}{\sin 6x} &= \lim_{x \rightarrow 0} \frac{6x}{2 \sin 6x} && \text{Multiplying top and bottom by 2} \\ &= \frac{1}{2} \lim_{x \rightarrow 0} \frac{6x}{\sin 6x} \\ &= \frac{1}{2}(1) && \text{Applying rule (ii) above} \\ &= \frac{1}{2}. \end{aligned}$$

Example 3

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{3x}{\tan 4x} &= \lim_{x \rightarrow 0} \frac{3x}{\frac{\sin 4x}{\cos 4x}} \\ &= \lim_{x \rightarrow 0} \frac{3x \cos 4x}{\sin 4x} \\ &= 3 \lim_{x \rightarrow 0} (\cos 4x) \left(\frac{x}{\sin 4x} \right) \\ &= 3 \left(\lim_{x \rightarrow 0} \cos 4x \right) \left(\lim_{x \rightarrow 0} \frac{x}{\sin 4x} \right) \\ &= 3 \left(\lim_{x \rightarrow 0} \cos 4x \right) \left(\frac{1}{4} \lim_{x \rightarrow 0} \frac{4x}{\sin 4x} \right) \\ &= \frac{3}{4}. \end{aligned}$$

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Try the following exercises for practice:

(a)

$$\lim_{x \rightarrow 0} \frac{\sin 7x}{x}$$

(b)

$$\lim_{x \rightarrow 0} \frac{x}{\sin 8x}$$

(c)

$$\lim_{x \rightarrow 0} \frac{\sin 7x}{3x}$$

(d)

$$\lim_{t \rightarrow 0} \frac{\tan t}{t}$$

(e)

$$\lim_{\theta \rightarrow 0} \frac{5 \sin \theta}{\theta}$$

Solutions

(a) 7

(b) $\frac{1}{8}$

(c) $\frac{7}{3}$

(d) 1

(e) 5