p-SERIES II (OF II)

Limit of a p-Series

There are two possible outcomes for the limit of a p-series.

$$\text{(1) If } p > 1, \quad \text{then} \quad \sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges.}$$

$$\text{(2) If } p \leq 1, \quad \text{then} \quad \sum_{n=1}^{\infty} \frac{1}{n^p} = \infty.$$

That means, if $p > 1$, then the p-series converges. However there is no formula telling us what the explicit limit is (like we have in the case of the geometric series). If $p \leq 1$ then the p-series diverges to plus infinity.

Example 2.1: Let $p = 1$. In this case $p \leq 1$ and hence

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \sum_{n=1}^{\infty} \frac{1}{n} = \infty,$$

that is, this p-series (the harmonic series) diverges to infinity, (see also Example 1.3 on the handout Integral Test I).

Example 2.2: Let $p = 2$. In this case $p > 1$ and hence

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \sum_{n=1}^{\infty} \frac{1}{n^2},$$

converges but we do not know what its limit is.

Example 2.3: Let $p = -2$. In this case $p \leq 1$ and hence

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \sum_{n=1}^{\infty} n^2 = \infty,$$

that is, this p-series diverges to infinity. This result is also obvious if we take a look at the sequence of partial sums, which is $s_1 = 1$, $s_2 = 5$, $s_3 = 14$, $s_4 = 30$ and so on, (see also Example 1.3 on the handout p-Series I). Clearly a sequence that increases so strongly tends towards infinity.

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