p-SERIES I (OF II)

Definition of a p-Series

Let p be a real number. Then we can consider the following sequence of real numbers

$$1, \quad \frac{1}{2^p}, \quad \frac{1}{3^p}, \quad \frac{1}{4^p}, \quad \ldots$$

This sequence can be expressed in a more compact form as

$$\left(\frac{1}{n^p}\right)_{n\geq 1}$$

Its corresponding series is expressed by

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

and we call this series a **p-series**. The elements of the p-series are:

$$\begin{aligned}
s_1 &= 1\\
s_2 &= 1 + \frac{1}{2^p}\\
s_3 &= 1 + \frac{1}{2^p} + \frac{1}{3^p}\\
s_4 &= 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p}\\
&\vdots\\
s_k &= 1 + \frac{1}{2^p} + \frac{1}{3^p} + \ldots + \frac{1}{k^p}, \quad \text{for } k \geq 1
\end{aligned}$$

Note that the elements $\frac{1}{n^p}$ are strictly positive, for all $n \geq 1$ and all p. As $s_{k+1} = s_k + a_{k+1}$ (see the handout What is a Series I), we can tell that the sequence $(s_k)_{k\geq 1}$ of partial sums is strictly increasing. That means, there are only two possibilities for the limit of a p-series, either it has a finite limit or it diverges to plus infinity.

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Examples of p-Series

For every real number p we get a different p-series. That means a p-series is uniquely determined by the value of p.

Example 1.1: Let $p = 1$. Then the sequence $\left(\frac{1}{n^p}\right)_{n\geq 1}$ becomes $\left(\frac{1}{n}\right)_{n\geq 1}$, and it consists of the elements

$$1, \quad \frac{1}{2}, \quad \frac{1}{3}, \quad \frac{1}{4}, \quad \ldots$$

Now the corresponding p-series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

has the elements

$$s_1 = 1, \quad s_2 = \frac{3}{2}, \quad s_3 = \frac{11}{6}, \quad s_4 = \frac{25}{12}, \quad \ldots$$

This p-series (that is, when $p = 1$) is also known as the **harmonic series**.

Example 1.2: Let $p = 2$. Then the sequence $\left(\frac{1}{n^p}\right)_{n\geq 1}$ becomes $\left(\frac{1}{n^2}\right)_{n\geq 1}$, and it consists of the elements

$$1, \quad \frac{1}{4}, \quad \frac{1}{9}, \quad \frac{1}{16}, \quad \ldots$$

The corresponding p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

has the elements

$$s_1 = 1, \quad s_2 = \frac{5}{4}, \quad s_3 = \frac{49}{36}, \quad s_4 = \frac{205}{144}, \quad \ldots$$

Example 1.3: Let $p = -2$. Then the sequence $\left(\frac{1}{n^p}\right)_{n\geq 1}$ becomes $(n^2)_{n\geq 1}$, and it consists of the elements

$$1, \quad 4, \quad 9, \quad 16, \quad \ldots$$

The corresponding p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^{-2}} = \sum_{n=1}^{\infty} n^2$$

has the elements

$$s_1 = 1, \quad s_2 = 5, \quad s_3 = 14, \quad s_4 = 30, \quad \ldots$$